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ON THE CENTRALIZER OF INVARIANT FUNCTIONS ON A HAMILTONIAN G-SPACE

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Abstract

A Hamiltonian action of a Lie group G on a symplectic manifold M gives rise to a moment map $J: M \to \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual space to the Lie algebra $\mathfrak{g}$ of G. The functions on M that are pullbacks of functions on $\mathfrak{g}^*$ by the moment map form a Poisson subalgebra of $C^\infty(M)$. Such functions are called *collective*. Assume that G and M are compact and connected. It is easy to see that the centralizer of collective functions in $C^\infty(M)$ consists of G-invariant functions. It was conjectured by Guillemin and Sternberg in [3] that the converse is also true, namely that the centralizer of the invariants is the set of collective functions. The main result of this paper is the proof of the conjecture in the case where the image of the moment map misses the walls of Weyl chambers in $\mathfrak{g}^*$, i.e., when the stabilizers under the coadjoint action of the points in $J(M)$ are all tori. An example shows that if $J(M)$ intersects the walls, the conjecture may fail.

Introduction

Let G be a compact connected Lie group acting on a compact connected symplectic manifold M with an (equivariant) moment map $J: M \to \mathfrak{g}^*$. A function f on M is said to be *collective* if it is a pullback by the moment map of a smooth function ϕ on $\mathfrak{g}^*$, $f = \phi \circ J$.

$C^\infty(M)$ and $C^\infty(\mathfrak{g}^*)$ are Lie algebras under Poisson bracket. In addition the Poisson bracket has the derivation property,

$$\{f, f_1 f_2\} = \{f, f_1\} f_2 + f_1 \{f, f_2\}.$$

The map $J^*: C^\infty(\mathfrak{g}) \to C^\infty(M)$, $\phi \to J^*\phi$ induced by J is a morphism of Poisson algebras. Thus collective functions form a subalgebra of $C^\infty(M)$. Another subalgebra of $C^\infty(M)$ is the set of G-invariant functions $C^\infty(M)^G$.

The two subalgebras are related. For a subset A of $C^\infty(M)$ define the *centralizer* of A to be the set of all functions that Poisson commute with the functions in A, and denote it by A^c. (Note that in [3] A^c is called the commutant of A.) A^c is a subalgebra for any A. The centralizer of

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the collective functions is the set of invariants. Indeed, if $\xi \in \mathfrak{g}$, then $\xi \in C^\infty(\mathfrak{g}^*)$ and $\{\xi \circ J, f\} = \xi_M(f)$, where ξ_M is the vector field on M induced by ξ.

Guillemin and Sternberg conjectured in [3] that the converse is true, namely that the centralizer of the invariants consist of the collective functions,

$$(C^\infty(M)^G)^c = J^* C^\infty(\mathfrak{g}^*). \tag{$*$}$$

They proved the conjecture in two special cases. Clearly the collective functions are contained in $(C^\infty(M)^G)^c$. There are two obstructions for the equality to hold.

The first one is the connectedness of the level sets of the moment map. Indeed, let c be a regular value of the moment map J. By Sard's theorem there exists a neighborhood U of c in $\mathfrak{g}^*$ consisting of regular values. Since J is proper, $J: J^{-1}(U) \to U$ is a fibration. Suppose $J^{-1}(c)$ is not connected. Let W_1, W_2 be components of $J^{-1}(U)$. Consider two smooth functions ϕ_1, ϕ_2 on $\mathfrak{g}^*$ supported in U. Let $f = J^*\phi_1|W_1 + J^*\phi_2|W_2$. Then f is a smooth function on M, it commutes with the invariants but it is not collective. Note that for a torus the level sets of the moment map are always connected [1].

The second obstruction is the singularities of the moment map. A function $f \in (C^\infty(M)^G)^c$ may be of the form $J^*\phi$ but ϕ need not be smooth. Consider for example the action of SU(2) on $M = \mathbf{C}^2$. In this case $C^\infty(M)^{\mathrm{SU}(2)}$ consists of the functions of $|z|^2$. So $(C^\infty(M)^{\mathrm{SU}(2)})^c$ contains $|z|^2$, but $|z|^2$ is not collective. Note however that $(|z|^2)^2$ *is* contained in $J^*C^\infty(\mathfrak{su}(2)^*)$, and that $J^*C^\infty(\mathfrak{su}(2)^*)$ together with $|z|^2$ generate $(C^\infty(M)^{\mathrm{SU}(2)})^c$. In the example the manifold $\mathbf{C}^2$ is not compact. However the action SU(2) on $\mathbf{C}^2$ can be easily "compactified" to an action of SU(2) on $\mathbf{C}P^2$.

Under certain restrictions on the image of the moment map the equality $(*)$ holds.

Theorem 1. *Let G, M, $J: M \to \mathfrak{g}^*$ and $C^\infty(M)^G$ be as above. If $J(M)$ misses the walls of the Weyl chambers in $\mathfrak{g}^*$, i.e., if stabilizers of points in $J(M)$ are all tori, then the centralizer of the invariants is collective,*

$$(C^\infty(M)^G)^c = J^* C^\infty(\mathfrak{g}^*).$$

We shall reduce the proof of Theorem 1 to that of a special case, namely,

Theorem 2. *Let G be a torus. Then the centralizer of the invariants is collective, i.e., $(C^\infty(M)^G)^c = J^*C^\infty(\mathfrak{g}^*)$.*

Theorem 1 does not settle the issue completely. It may very well happen that the image contains bad points and yet $(*)$ still holds.

The strategy of the proof is as follows. First, using local normal form theorems for the moment map ([4], [5], [6]) we explicitly compute the action and the moment map in a neighborhood of an orbit. Then we show that $(*)$ holds in such a neighborhood. Using connectedness of level sets of toral moment maps we show that $(*)$ holds in a neighborhood of a level set. A partition of unity argument finishes the proof. Finally we prove that Theorem 2 implies Theorem 1.

A few useful facts

We recall a few facts about the moment map. By the definition of the moment map we have

$$(\spadesuit) \qquad \langle dJ_p(v), \xi\rangle = [i(\xi_M(p))\omega_p](v),$$

where $p \in M$, $v \in TM_p$, $\xi \in \mathfrak{g}$, and ξ_M denotes the vector field induced on M by ξ. The set $\{\xi_M(p) : \xi \in \mathfrak{g}\}$ is the tangent space to the G-orbit through p. Hence

Proposition 1. *The symplectic perpendicular to the tangent space to the orbit at p is the kernel of dJ_p. The image of dJ_p is the annihilator of the Lie algebra of the stabilizer group of p. In particular, dJ_p is surjective if and only if the stabilizer of p is discrete.*

Suppose $J(p)$ in $\mathfrak{g}^*$ is fixed by G. Let ξ, η be vectors in $\mathfrak{g}$. Since J is equivariant, $dJ_p(\eta_M(p)) = N_{\mathfrak{g}^*}(J(p))$. Since $J(p)$ is fixed, the vector field $\eta_{\mathfrak{g}^*}$ induced by the coadjoint action is zero at p. By $(\spadesuit)$

$$0 = \langle \eta_{\mathfrak{g}^*}(J(p)), \xi\rangle = \omega_p(\xi_M(p), \eta_M(p)).$$

We have proved

Lemma 1. *If $J(p)$ is a fixed point of the coadjoint action of G, then the orbit $G \cdot p$ is isotropically embedded in M.*

One more preliminary remark. Let N be any manifold, P be a Poisson manifold and f be a function on P. Then we may consider the Hamiltonian vector field Ξ_f of f on P to be a vector field on $N \times P$. Alternatively, we can make $N \times P$ into a Poisson manifold by taking the bracket to be zero on N, and then consider the Hamiltonian vector field of f, which is also a smooth function on $N \times P$, with respect to the Poisson structure on $N \times P$.

Proof of Theorem 2

We now begin proving Theorem 2. Till the end of the proof, assume that G is a torus. Then G-orbits are isotropically embedded. In [4] and

[5] Guillemin and Sternberg give a recipe for computing the moment map near an isotropic orbit. C.-M. Marle gives a similar recipe in [6]. The computation relies on Weinstein's isotropic embedding theorem.

Theorem (Isotropic embedding). *Any isotropic embedding $i\colon X \to M$ defines a symplectic normal bundle $N(i) \to X$ where $N(i) = TX^{\perp}TX$. If f is a symplectic diffeomorphism defined on some neighborhood of $i_1(X)$ and satisfying*

$$i_2 = f \circ i_1,$$

where i_1, i_2 are isotropic embeddings of X, then f induces a symplectic isomorphism, $L_f\colon N(i_1) \to N(i_2)$ of the corresponding symplectic normal bundles. Conversely, given any symplectic isomorphism $L\colon N(i_1) \to N(i_2)$ there exists an f with $L = L_f$.

In a presence of a compact group K of automorphisms, all of the above assertions are true in the category of K morphisms.

Let $Z = G{\cdot}p$ be an orbit, $i\colon Z \to M$ be its embedding, $N(i) = TZ^{\perp}/TZ$ be its symplectic normal bundle, $V = TZ_p^{\perp}/TZ_p$ be a typical fiber of $N(i)$, and S be the stabilizer of p. S is a compact abelian subgroup of G, and G is a principal S-bundle over Z. S acts symplectically on V. We have a surjection

$$G \times V \to N(i), \qquad (g, v) \to g \cdot v.$$

It is easy to see that the map factors through an isomorphism of symplectic G-bundles $G \times_S V \to N(i)$.

We construct a Hamiltonian G-space Y and an isotropic embedding j of Z into this space so that $N(j)$ is isomorphic to $N(i)$ as a symplectic G-vector bundle. Then the embedding $j\colon Z \to Y$ serves as a model for $i\colon Z \to M$. More precisely, by the isotropic embedding theorem there exist neighborhoods U_i and U_j of $i(Z)$ and $j(Z)$ respectively and a G-equivariant symplectic diffeomorphism $\Psi\colon U_i \to U_j$ so that $j = \Psi \circ i$.

Suppose first that S is connected. Then S is a torus and $G \approx K \times S$ for some torus K; K is isomorphic to Z. Then

$$G \times_S V = (K \times S) \times_S V \approx K \times V.$$

Here $G = K \times S$ acts on $K \times V$ as follows: K acts on K by the multiplication on the right and trivially on V, S acts symplectically on V and trivially on K. Consider $X = T^*K \times V$. Identify T^*K with $\mathfrak{k}^* \times K$. K acts symplectically on T^*K with a moment map Φ_K equal projection on the first factor. S acts symplectically on V with the resulting moment map $\Phi_S\colon V \to \mathfrak{s}^*$. We see that X is a Hamiltonian G space with a moment map

$$\Phi^X = (\Phi_K, \Phi_S)\colon T^*K \times V \to \mathfrak{k}^* \times \mathfrak{s}^* \approx \mathfrak{g}^*.$$

Now $j: K \to T^*K \times V$ is a model for $i: Z \to M$. Indeed it is enough to observe that $N(j) \approx K \times V$ and that the action of G on $N(j)$ is the right one.

In general let S_0 be the identity component of S. Then $S = D \times S_0$ for some discrete subgroup D of G. Choose a torus K containing D so that $G = K \times S_0$. Then the orbit Z is isomorphic to K/D, and the symplectic normal bundle $N(i)$ is isomorphic to $(K \times V)/D \equiv K \times_D V$. The action of G on $K \times V$ projects to an action of G on $(K \times V)/D$. Consider the manifold $Y = (T^*K \times V)/D$. It is a Hamiltonian G-space and the moment map Φ^Y makes the diagram

$$\begin{array}{ccc} X = T^*K \times V & \xrightarrow{\Phi^X} & g^* \\ \downarrow \pi & \nearrow \Phi^Y & \\ Y = (T^*K \times V)/D & & \end{array}$$

commute. Y is a vector bundle over K/D, so Z embeds in Y as zero section. Moreover the symplectic normal bundle of the embedding $j': Z \to Y$ is isomorphic to $(K \times V)/D$. In other words $j': Z \to Y$ is a model for $i: Z \to M$.

We use the model to prove

Lemma 2 (*Centralizer of invariants is locally collective*). *If a function f commutes with all G-invariant functions on M, i.e., if $f \in (C^\infty(M)^G)^c$, then for any point p in M there exist an open set W in M containing p and a function ϕ in $C^\infty(\mathfrak{g}^*)$ so that $f \mid W = \phi \circ J \mid W$.*

But first we need two rather technical results. Let V be a symplectic vector space, S be a compact abelian group acting symplectically on V and S_0 be the identity component of S. Then $S = D \times S_0$ for some discrete group D. The first result that we need is that if a function $f \in C^\infty(V)$ Poisson commutes with all S_0-invariant functions on V, then it commutes with all S-invariant functions as well. This is a special case of the following proposition.

Proposition 2. *Let K be a compact Lie group and D be a finite group acting on a Poisson manifold M. Assume further that the Poisson actions of K and D commute. If f is a function in $C^\infty(M)$ that commutes with all $K \times D$ invariant functions on M, then for any $h \in C^\infty(M)^K$*

$$\{f, h\} = 0.$$

Proof. The main idea is to show that for a generic point x in M and for any function h in $C^\infty(M)^K$ there exists a function h_D in $C^\infty(M)^{K\times D}$ so that $h \equiv h_D$ near x. For then we have

$$\{f, h\}(x) = \{f, h_D\}(x) = 0.$$

The construction of such a function h_D relies on the theorem about the principal orbit type. We need to recall a few facts concerning the actions of compact groups. See, for example, [2] for proofs and historic references.

Given a subgroup H of K denote by (H) the class of subgroups of K that are conjugate to H.

Proposition A. *Let H be a stabilizer subgroup of K. Then*

$$M_{(H)} = \{x \in M : \text{stabilizer of } x \text{ is conjugate to } H\}$$

is a submanifold of M (which may have components of different dimensions).

Proposition B (*principal orbit type*). *There exists a unique stabilizer type (H), the principal orbit type, such that $M_{(H)}$ is open and dense. Each other stabilizer type H' satisfies $(H) < (H')$, i.e., H is subconjugate to H'.*

Proposition B implies that for any x in $M_{(H)}$ the normal bundle to the orbit $K \cdot x$ is trivial. Hence the orbit space $M_{(H)}/K$ is a manifold, and $M_{(H)}$ fibers over it with a typical fiber being K/H, $\pi \colon M_{(H)} \to M_{(H)}/K$. The action of D preserves $M_{(H)}$ and descends to an action on $M_{(H)}/K$. For some normal subgroup D' of D, D/D' acts effectively on $M_{(H)}/K$. Moreover, since D/D' is finite there exists a dense open subset W of $M_{(H)}/K$ on which the action is free. Let $M_0 = \pi^{-1}(W)$; M_0 is a dense open subset of M, the set of generic points in the statement of the proposition. Note that any K-invariant function on M_0 is D'-invariant.

For x in M_0 consider $\pi(x)$. Since the action of D/D' on W is free, there exists a neighborhood U of $\pi(x)$ so that

$$dD' \cdot U \cap U = \varnothing \quad \text{for } dD' \neq D'.$$

Choose a bump function τ' supported in U with $\tau' \equiv 1$ near $\pi(x)$, and set $\tau = \tau' \circ \pi$. Then τ is K-invariant and, for any y near x, $\tau(d \cdot y)$ is 1 if $d \in D'$ and 0 otherwise. Now for h in $C^\infty(M)^K$ set

$$h_D(y) = \frac{1}{|D'|} \sum_{d \in D} (\tau h)(d \cdot y).$$

The function h_D is $K \times D$ invariant and is equal to h near x. This proves Proposition 2.

In [2] Guillemin and Sternberg proved that for a toral action on a vector space the centralizer of the invariant functions is collective. An examination of the proof shows that a somewhat more general statement holds.

Proposition 3 (*c.f. Proposition* 4.1 *in* [4]). *Let N be any manifold and $S \to \mathrm{Sp}(W)$ be a symplectic representation of a torus S. (Then the action of S on W is Hamiltonian with the moment map $J \colon W \to \mathfrak{s}^*$ given by*

$2\langle \xi, J(v)\rangle = \omega(\xi v, v)$ for $v \in V$, $\xi \in \mathfrak{s} \to \mathfrak{sp}(W)$. Here ω is the symplectic form on W.) $N \times W$ is a Poisson manifold with the bracket being zero on N. If a function $f \in C^\infty(N \times W)$ Poisson commutes with all S-invariant functions on W, then there exists a smooth function ϕ on $N \times \mathfrak{s}^*$ so that $f(m, v) = \phi(m, J(v))$.

Proof of Lemma 2. Since a small enough neighborhood of the orbit Z in M is equivalent to a neighborhood of Z in the model space Y, it suffices to prove the lemma for the functions on Y. We need to show that if a function on Y commutes with all G-invariant functions, then it is collective. For that it is enough to prove that its pullback to the covering space X is collective. Let f be such a pullback. We will show that there exists a function ϕ in $C^\infty(\mathfrak{g}^*)$ such that at any point $(\eta, a, v) \in \mathfrak{k}^* \times K \times V = X$

$$f(\eta, a, v) = \phi(\eta, J^V(v)),$$

where $J^V \colon V \to \mathfrak{s}^*$ is the moment map on V coming from the action of S_0.

Since the covering map $\pi \colon X \to Y$ is a Poisson map, f commutes with pullbacks of G-invariant functions on $Y = \mathfrak{k}^* \times (K \times_D V)$. That is, f commutes with all G-invariant functions on X that are also D-invariant.

Consider a vector ξ in $\mathfrak{k}$. We can think of it as a smooth G- and D-invariant function on X. Hence its bracket with f is zero. But $\{f, \xi\}$ being zero for all ξ in $\mathfrak{k}$ implies that f is K-invariant. Thus f is constant along K. f also commutes with all S-invariant functions on V. By Proposition 2, f commutes with S_0-invariant functions. (Note that if S is discrete, then K is simply G and Proposition 2 says that f is constant on V. So f is a function of $\mathfrak{k}^*$ alone, and this proves Lemma 2 in the special case.)

In general by Proposition 3 there exists a smooth function ϕ on $(\mathfrak{k}^* \times K) \times \mathfrak{s}^*$ so that $f(\eta, a, v) = \phi(\eta, a, J^V(v))$. But f does not depend on a, so ϕ is a smooth function on $\mathfrak{k}^* \times \mathfrak{s}^* = \mathfrak{g}^*$.

This concludes the proof of Lemma 2.

Remarks. 1. Note that Lemma 2 holds with parameters. Let N be a manifold and Y be as above. Extend the Poisson bracket on Y to $N \times Y$ by zero. Suppose that f in $C^\infty(N \times Y)$ has the property that $\{h, f\} = 0$ for any function h in $C^\infty(Y)^G$. Then there exists a function $F \in C^\infty(N \times Y)$ so that

$$f(p, y) = F(p, \Phi^Y(y))$$

for $(p, y) \in N \times Y$.

We shall need this observation to prove Theorem 1.

2. The image of Y under the moment map Φ^Y is a linear cone

$$C = \mathfrak{k}^* \times \{\beta : \beta = \sum t_j \kappa_j, t_j \geq 0\} \subset \mathfrak{k}^* \times \mathfrak{s}^* = \mathfrak{g}^*.$$

So under the moment map $J: M \to \mathfrak{g}^*$ the image of a sufficiently small neighborhood of the orbit $G \cdot p$ is of the form $(\alpha + C) \cap W$, where W is an open set in $\mathfrak{g}^*$ about $\alpha = J(p)$. More generally we have

Lemma 3. *There exist a neighborhood U of the level set $J^{-1}(\alpha)$ whose image under J is of the form $(\alpha + C) \cap W'$ where W' is an open set in $\mathfrak{g}^*$ about α and C is as above.*

PROOF. We know that the image of a small neighborhood of a point in $J^{-1}(\alpha)$ is an open subset of *some* cone C' translated to α. The point of the lemma is that the cone does not vary along the level set. This follows from two observations. First of all $J^{-1}(\alpha)$ is connected, so it is enough to show that the cone does not vary locally. But the local behavior is modeled by (Y, Φ^Y), and along the zero level set of Φ^Y the cone does not vary. Therefore for any q in $J^{-1}(\alpha)$ there exist open sets U_q in M containing q and W_q in $\mathfrak{g}^*$ containing α so that $J(U_q) = (\alpha + C) \cap W_q$. Since $J^{-1}(\alpha)$ is compact, there exist $q_1, \cdots, q_L$ in $J^{-1}(\alpha)$ such that the corresponding sets $U_1, \cdots, U_L$ cover $J^{-1}(\alpha)$. The lemma now follows with $W' = W_1 \cap W_2 \cap \cdots \cap W_L$.

Remark. We see from the proof of the lemma that $J \mid U$ is an open map into the translated cone $\alpha + C$. So for any open set $U_0 \subset U$ there exists an open set W_0' in $\mathfrak{g}^*$ such that $J(U_0) = (\alpha + C) \cap W_0'$.

Our next step is to improve on Lemma 2. We show the centralizer of invariants is collective not just in a neighborhood of an orbit but in a neighborhood of the whole level set.

Lemma 4. *If f commutes with all G-invariant functions on M, i.e., if $f \in (C^\infty(M)^G)^c$, then for any α in $\mathfrak{g}^*$ there exist an open set U_α containing $J^{-1}(\alpha)$ and a function ϕ in $C^\infty(\mathfrak{g}^*)$ so that*

$$f \mid U_\alpha = \phi \circ J \mid U_\alpha.$$

PROOF. Choose an open set U in M about $J^{-1}(\alpha)$ as in Lemma 3. Then $J(U) = (\alpha + C) \cap W'$ for some open set W' in $\mathfrak{g}^*$. By Lemma 2 for any q in $J^{-1}(\alpha)$ there exist an open set $U(q)$ and a function ϕ_q in $C^\infty(\mathfrak{g}^*)$ so that $f \mid U(q) = \phi_q \circ J \mid U(q)$. We may assume that $U(q) \subset U$ for all q. Since $J^{-1}(\alpha)$ is compact, we can cover $J^{-1}(\alpha)$ by finitely many $U(q)$, say $U_1 = U(q_1), \cdots, U_L = U(q_L)$. Let $\phi_1, \cdots, \phi_L$ be the corresponding functions in $C^\infty(\mathfrak{g}^*)$. Since $J^{-1}(\alpha)$ is connected, we may assume that $U_i \cap U_{i+1}$ is nonempty for $i = 1, \cdots, L$.

We proceed by induction on L. Suppose for simplicity that L is 2, so $J^{-1}(\alpha) \subset U_1 \cap U_2$. Now $f \mid U_i = \phi_i \circ J \mid U_i$ for $i = 1, 2$ implies that

$\phi_1 \mid J(U_1 \cap U_2) = \phi_2 \mid J(U_1 \cap U_2)$. By the remark after Lemma 3 there exists an open set W'' in $\mathfrak{g}^*$ so that $J(U_1 \cap U_2) = (\alpha + C) \cap W''$. Let $\phi = \phi_1$, and $U_\alpha = J^{-1}(W'') \cap (U_1 \cap U_2)$. Then $f \mid U_\alpha = \phi \circ J \mid U_\alpha$.

It is clear now how the induction works in general.

A simple partition of unity argument finishes the proof of Theorem 2. For a point α in $J^{-1}(\alpha)$ there exist an open set U_α in M, $J^{-1}(\alpha) \subset U_\alpha$, an open set $W(\alpha)$ in $\mathfrak{g}^*$, a cone C_α and a function $\phi_\alpha \in C^\infty(\mathfrak{g}^*)$ so that

(i) $J(U_\alpha) = (\alpha + C_\alpha) \cap W(\alpha)$ and

(ii) $f \mid U_\alpha = \phi_\alpha \circ J \mid U_\alpha$.

$J(M)$ is compact, hence there exist $\alpha_1, \cdots, \alpha_s$ in $\mathfrak{g}^*$ such that $W_1 = W(\alpha_1), \cdots, W_s = W(\alpha_s)$ cover $J(M)$. Let $\phi_1, \cdots, \phi_s$ be the corresponding functions in $\mathfrak{g}^*$. Let W_0 be the compliment of $J(M)$ in $\mathfrak{g}^*$, i.e., let $W_0 = \mathfrak{g}^* \backslash J(M)$. Choose a partition of unity $\{\rho_0, \cdots, \rho_s\}$ on $\mathfrak{g}^*$ subordinate to $\{W_0, \cdots, W_s\}$; then $\{J^*\rho_1, \cdots, J^*\rho_s\}$ is a partition of unity on M subordinate to $\{U_1, \cdots, U_s\}$. Let $\phi = \sum \rho_i \phi_i$. Then $J^*\phi = \sum J^*\rho_i J^*\phi_i = \sum (J^*\rho_i) f = f$. This finishes the proof of Theorem 2.

Corollary. *Let N be a manifold, T a torus and P a Hamiltonian T-space with moment map $\Phi: P \to \mathfrak{t}^*$. Extend the Poisson bracket on P to $N \times P$ by zero. Suppose h in $C^\infty(N \times P)$ Poisson commutes with all functions in $C^\infty(P)^T$. Then there exists a function $\mathfrak{h}$ in $C^\infty(N \times \mathfrak{t})$ so that*

$$h(n,p) = \mathfrak{h}(n, \Phi(p)).$$

Proof of Theorem 1

We now proceed with the proof of Theorem 1. Recall that G is a compact connected Lie group acting on a compact connected symplectic manifold M in a Hamiltonian fashion with a moment map $J: M \to \mathfrak{g}^*$. Put a G-invariant metric on $\mathfrak{g}^*$, and use it to identify $\mathfrak{g}^*$ with $\mathfrak{g}$. Let $\mathfrak{g}_{\mathrm{reg}}$ be the set of regular elements of $\mathfrak{g}$, i.e., if

$$\mathfrak{g}_{\mathrm{reg}} = \{\xi \in \mathfrak{g}: \text{ stabilizer of } \xi \text{ is a torus}\}.$$

By assumption $J(M)$ is contained in $\mathfrak{g}_{\mathrm{reg}}$. Fix a maximal torus T in G, let $\mathfrak{t}$ be its Lie algebra and let R be a connected component of $\mathfrak{t} \cap \mathfrak{g}_{\mathrm{reg}}$. G is a principal T-bundle over G/T. The map

$$G \times R \to \mathfrak{g}_{\mathrm{reg}}, \qquad (g, \xi) \to \mathrm{Ad}_g(\xi)$$

is a surjection. It induces a G-equivariant bijection $G \times_T R \to \mathfrak{g}_{\mathrm{reg}}$. Here $G \times_T R$ denotes the associated fiber bundle over G/T with fiber R. The moment map J is transversal to R, so $F = J^{-1}(R)$ is a submanifold of

M. Moreover F is symplectic. (See Theorem 26.7 in [4].) Since the inverse image of R under J equals the inverse image of its closure, F is closed. The fact that $M = J^{-1}(\mathfrak{g}_{\mathrm{reg}})$ and equivariance of J imply that M is diffeomorphic to $G \times_T F$ as a G-space. More explicitly let

$$p_1 \colon G \times F \to G \times_T F$$

be the projection. Then the map $G \times_T F \to M$ is given by $p_1(g, f) \to g \cdot f$. It is a well-known fact that G/T is simply connected. Since M is a connected fiber bundle with a simply connected base its fiber F is connected.

Let $j = J \mid F$. Then F is a Hamiltonian T-space, and j is a corresponding T-moment map. The map $\mathrm{id} \times j \colon G \times F \to G \times R$ induces a G-equivariant map of fiber bundles $G \times_G F \to G \times_T R$. Since J is also G-equivariant the induced map equals J.

Let $p_2 \colon G \times R \to G \times_T R$ be the projection. Then the map $G \times_T R \to \mathfrak{g}_{\mathrm{reg}}$ is given by $p_2(g, r) \to g \cdot r$. Let $\mu \colon U \to G$ be a local section of $G \to G/T$. μ induces trivializations of $\pi_1 \colon G \times_T F \to G/T$ and $\pi_2 \colon G \times_T R \to G/T$:

$$\begin{aligned} \phi_1 \colon U \times F \to G \times_T F, \qquad & (u, q) \to p_1(\mu(u), q), \\ \phi_2 \colon U \times R \to G \times_T R, \qquad & (u, r) \to p_2(\mu(u), r). \end{aligned}$$

Now

$$\begin{aligned} J(\phi_1(u, q) = J(p_1(\mu(u), q)) = J(\mu(u) \cdot q) = \mu(u) J(q) \\ = \mu(u) j(q) = \phi_2(\mu(u), j(q)). \end{aligned}$$

Thus, with respect to the identifications, $J \mid \pi_1^{-1}(U) \colon \pi_1^{-1}(U) \to \pi_2^{-1}(U)$ is given by $J(u, q) = (u, j(q))$.

Since $M = G \times_T F$ there exists a 1-1 correspondence between G-invariant functions on M and T-invariant functions on F. In one direction the correspondence is simply restriction to the fiber. In the other direction, a T-invariant function on F pulls up to a G- and T-invariant function on $G \times F$ and so descends to a G-invariant function $G \times_T F$.

This carries over to the correspondence between Hamiltonian vector fields. (Recall that F is a symplectic submanifold.) That is, given a G-invariant function f, restricting it to F and taking the Hamiltonian vector field of the restriction is the same as taking the Hamiltonian vector field Ξ_f of f and restricting it to F. To prove this it is enough to show that Ξ_f is tangent to F. So let p be a point in $F \to G \times_T F$. Since f is constant along the orbit $G \cdot p$, $\Xi_f(p)$ lies in the symplectic perpendicular $T_p(G \cdot p)^{\perp}$. By Proposition 1 we have $T_p(G \cdot p)^{\perp} = \ker dJ_p$. But $F = J^{-1}(R)$ and J

intersects R transversely. Hence T_pF contains $\ker dJ_p$ and therefore $\Xi_f(p)$ lies in T_pF.

Consider h in $(C^\infty(M)^G)^c$. Assume for a moment that the support of h is contained in $\pi_1^{-1}(U)$ and that $G \to G/T$ is trivial over U. Then $\pi_1^{-1}(U) = U \times F$, and it follows from the discussion above that h is killed by the Hamiltonian vector fields of the T-invariant fucntions on F. By Corollary to Theorem 2, there exists a function $\mathfrak{h}$ in $C^\infty(U \times \mathfrak{t})$ so that $h(u,f) = \mathfrak{h}(u, j(f))$. Thus $H = J^*\phi$ for some ϕ in $C^\infty(g)$ $(= C^\infty(\mathfrak{g}^*))$.

In general let $\{U_i\}$ be a cover of G/T such that the $G \mid U_i$ are trivial. Choose a partition of unity $\{\sigma_i\}$ subordinate to the cover. Then $\{\pi_i^*\sigma_i\}$ is a partition of unity on $G \times_T F$ and each $\pi_i^*\sigma_i$ is supported in $\pi_1^{-1}(U_i)$. Moreover, since $\pi_1 = \pi_2 \circ J$, $\pi_1^*\sigma_i$ are collective. Therefore if h is in $(C^\infty(M)^G)^c$, then $(\pi_1^*\sigma_i) \cdot h$ are also in $(C^\infty(M)^G)^c$. But by the discussion above $(\pi_1^*\sigma_i) \cdot h$ are collective, and so $h = \sum(\pi_i^*\sigma_i) \cdot h$ is also collective.

This finishes the proof of Theorem 1.

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